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Fundamental length from algebra

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Abstract

Advances in physics have required the application of more and more sophisticated mathematics. I present arguments supporting the contention that the next advance beyond quantum field theory will require the application of a non-associative algebra. The principle observable effect would be the appearance of a fundamental length. An experimental search for such an effect may be feasible.

PACS:

I. INTRODUCTION

When physics and astronomy first became quantitative disciplines, the number system employed was simply the real numbers. From Galileo through Newton and well into the 18th century C.E., the algebra of real numbers continued to suffice. With the advent of abstract wave phenomena, the advantages of using the algebra of complex numbers to describe phases became apparent. Although the application of groups to crystallography implicitly introduced non-commutative algebras, they did not become essential until the advent of quantum mechanics and non-Abelian gauge symmetries. These advances continued to suffice for quantum field theory with the caveat of renormalization, along with the expansion of geometry to $3+1$-dimensional space-time.

Separately, the study of gravity led to the inclusion of Riemanian geometries with their own mathematical complexities. Attempts to make further advances have included non-associative and non-commutative geometries, graded Lie algebras, also referred to as super-symmetric algebras, and associated with that, even higher space-time dimensions in string theory. [1, 2] My comments below do not have any apparent direct relation to these latter questions.

Instead, my focus is on the relation between the algebras in mathematics and the question of dimensional quantities in physics.

II. A FUNDAMENTAL SCALE

Einstein's $3+1$-dimensional space-time of special relativity [3] related spatial dimension and time by the universal constant of the speed of light, c. With the advent of the action principle approach to classical mechanics pursued for symmetries by Noether [4], and its application to quantum mechanics, Planck's universal constant [5] was recognized to be both related to a non-commutative algebra and to define a relation between momentum and spatial dimension. In effect, these advances reduced the variety of dimensionful physical quantities to just one, which may be taken (with the assistance of c) to be described equivalently as mass or length.

I turn next to the renormalization problem of quantum field theory. Although now wellregarded and viewed as an essential component of understanding the scale dependence of physical effects and quantities, it was initially viewed (by Feynman and others) as some sort of trick to avoid a fundamental question. By considering the nature of the associator compared with the nature of the commutator, we get an inkling of what that issue may be: A fundamental length that is a universal constant.

A. Commutator

The well-known quantum commutator between the spatial location operator, q, and the momentum operator, p, relates length and momentum to the Planck constant, $\hbar$, viz.

$$[q, p] = \imath\hbar \tag{1}$$

or in dimensional terms, thinking of p as some multiple ($\beta\gamma$ in special relativity) of mass (m) times the speed of light (c) affords the opportunity to reduce all quantities to the dimensions of length.

In quantum field theory, the commutator of scalar fields at two points, or equivalently a scalar field ($\phi(x)$) and its canonical momentum/spatial derivative ($\Pi_\phi(y)$) is similar (where x and y are spatial 3-vectors at the common time, $t = 0$):

$$[\phi(x), \Pi_\phi(y)]_{t=0} = \imath\hbar\delta^3(x - y). \tag{2}$$

So consistency requires the dimension of a scalar field be that of mass or, by the $\hbar$ equivalence, an inverse length as $\hbar$ (modulo c) reconciles the cubic dimension of mass on the lhs with the inverse dimensions of length provided by the δ-function on the rhs.

B. Associator

Similarly, if we consider the associator of three scalar fields,

$$\{\phi(x), \phi(y), \phi(z)\}_{t=0} = A\hbar\delta^3(x - y)\delta^3(y - z) \tag{3}$$

where A is some product of purely mathematical constants and a dimensionful universal constant, it follows that the dimension of A must be that of the cube of a length, ℓ_f, (or equivalently, the inverse of the cube of a mass). I propose that this defines a fundamental length. (The factor of $\hbar$ has been included to reconcile the dimensions of the fields and one of the δ-functions in what seems like a natural manner; eliminating it does change the number of powers of mass in the dimensionful constant, which may be significant.)

III. VALUE OF FUNDAMENTAL LENGTH

A value for a fundamental (Planck) length has been conjectured to be related to $\hbar$, and c and Newton's constant for gravity, G_N, as

$$\ell_{P\ell} = \sqrt{\frac{\hbar G_N}{c^3}} = 1.6 \times 10^{-35}\mathrm{m} \tag{4}$$

based on the concept that the corresponding "point" mass occupies a space smaller than that defined by its Schwarzschild radius and so must be a "black hole" such that anything entering that volume could not re-emerge. (It is difficult to see how Hawking radiation could arise from a point-particle black hole as opposed to one created initially from multiple components.) A similar l ength scale has been suggested on the basis of reconciling gravity and quantum mechanics [6], even in the absence of a well-defined theory of quantum gravity.

If we consider G_N in terms of a (Planck) mass,

$$G_N = \frac{\hbar c}{m_{P\ell}^2} \tag{5}$$

(where $m_{P\ell} \sim 2.2 \times 10^{-5}$g), this is reminiscent of the discussion of the classical electron radius, r_e, which relates the electrostatic self-interaction energy of a homogeneous charge distribution to the relativistic mass-energy of the electron, $m_e c^2$, namely,

$$r_{e,cl} = \alpha \frac{\hbar c}{m_e c^2} \tag{6}$$

where α is the fine structure constant of electromagnetism. However, this is much smaller than the quantum "size" of an electron at rest,

$$r_{e,qm} = \frac{\hbar c}{m_e c^2} \tag{7}$$

which does not include the factor of the fine structure constant, so that

$$r_{e,qm} >> r_{e,cl} \tag{8}$$

which has long been viewed as resolving the physical conundrum.

I suggest that, similarly, that the value of ℓ_f in the associator will satisfy

$$\ell_f >> \ell_{P\ell} \tag{9}$$

on the basis that there might well be something parallel to the fine structure constant included in G_N in a complete theory of quantum gravity.

However, there is as yet no data supporting any particular value. If the most massive particle known, the top quark, is indeed a "point" particle without substructure related to constituents of higher mass and shorter distance scale internal structure, we can only conclude that

$$\ell_f \lesssim 10^{-18}\mathrm{m} \tag{10}$$

which is still about 17 orders of magnitude larger than the Planck length. This bound may be reduced by an order of magnitude or two by experiments measuring top quark production or scattering and comparing with calculations that allow for deviations from point-like structure via a top quark form factor, if no deviation is observed within the experimental and theoretical uncertainties. I return to this issue in my conclusions.

IV. PHYSICS

A mathematical motivation for the choice of a non-associative algebra and the associator is straight-forward: This is the last of the classical algebras available, so it merits theoretical investigation in its own right.

However, there is a better physical motivation. The divergences in renormalizable quantum field theories that lead to the requirement of renormalization only develop with interactions, *i.e.*, where three or more fields meet at a point, not just two. A physical question to ask then is: What difference might there be if the order in which the fields are brought into contact differs: $x \to y$ at $z \neq x, y$ and $z \to x, y$ *vs.* $y \to z$ at $x \neq y, z$ and $x \to y, z$. (Time ordering may be essential to understanding this, which raises questions regarding avoiding frame dependence.) The difference in the various sequences of forming combinations is what is described mathematically by the associator.

For a black hole, the "no-hair" theorems ensure that there is no difference, *i.e.*, the associator must vanish. However, if $\ell_f > \ell_{P\ell}$, there may well be a difference that is physically relevant to field theory. Relativity complicates this all, somewhat, by making the size of separations frame dependent, but it is not unreasonable to consider that the center-of-momentum frame for three converging particles (from the three field operators) is the appropriate one in which to view the effects of ℓ_f.

V. EXPERIMENT AND APPLICATION

The possibility of a direct experimental search for effects of the Planck length has been suggested by Bekenstein [7, 8]. The feasibility has been examined by Maclay *et al.* [9] with the conclusion that while extremely difficult, the experiment may not be impossible. If the suggestion in this paper is valid, there is a motivation for embarking upon the search for the effect proposed by Bekenstein at every distance scale from the current limit down to the Planck length itself. An intermediate discovery may await an intrepid experimentalist.

If an experimental discovery is made to the effect that $\ell_f >> \ell_{P\ell}$, there may even be a bonus available to the theory of gravity. If we define the corresponding fundamental mass

$$m_f = \frac{\hbar}{\ell_f c} \tag{11}$$

then we might be able to identify

$$G_N = \zeta \frac{\hbar c}{m_f^2}. \tag{12}$$

Consistency requires that $\zeta << 1$ if $m_f << m_{P\ell}$. This allows for a possibility of describing why gravity is so weak (compared to other forces) in terms of some kind of effective coupling constant, although such a view might be at odds with General Relativity.

A. Other Applications

The associator and fundamental length offer opportunities to answer additional fundamental questions that have eluded resolution to the present.

These include a resolution of the question of the nature of the Coulomb potential for a "point" charge. A fundamental length would be expected to blunt the divergence at zero distance. A related issue is the electromagnetic self energy of such a charge and its contribution to the masses of the so-called fundamental particles. The Standard Model currently describes those masses in terms of the vacuum expectation value of the Higgs' field, but electromagnetic corrections are not ruled out by any means. Other (weak and strong interaction) self-energy contributions might also become calculable in a reliable and well-defined formulation.

In principle, this question should also apply to the gravitational self-energy of a "point" particle. Although quantum effects will spread the wave function of such a particle over a larger volume than that defined by the Schwarzschild radius for its mass, except for a particle of mass $m_{P\ell}$, the self-energy question appears in principle just as it does for the other fundamental interactions. This last point may, however, be obviated by Mottola's observations [10] of the quantum anomaly in the conservation equation for the divergence of the gravitational current.

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